SUBSEQUENCES OF NORMAL SEQUENCES

BY

TETURO KAMAE

ABSTRACT

In this paper, we characterize a set of indices $\tau = \{\tau(0) < \tau(1) < \cdots\}$ such that for *any* normal sequence $(\alpha(0), \alpha(1), \cdots)$ of a certain type, the subsequence $(\alpha(\tau(0)), \alpha(\tau(1)), \cdots)$ is a normal sequence of the same type. Assume that $\overline{\lim}_{n\to\infty} \tau(n)/n < \infty$. Then, we prove that τ has this property if and only if the 0-1 sequence $(\theta_\tau(0), \theta_\tau(1), \cdots)$, where $\theta_\tau(i) = 1$ or 0 according as $i \in \{\tau(j); j = 0, 1, \cdots\}$ or not, is *completely deterministic* in the sense of B. Weiss.

1. Introduction

Let Σ be any compact metric space. Let $N = \{0, 1, 2, \cdots\}$ be the set of non-negative integers. By Σ^N, we mean the product space of Σ with the product topology. The i-th coordinate $(i \in N)$ of $\alpha \in \Sigma^N$ is denoted by $\alpha(i)$. An element of Σ^N is called a *sequence*. Let T be the *shift* on Σ^N; $(T\alpha)(i) = \alpha(i+1)$ for any $\alpha \in \Sigma^N$ and $i \in N$.

By a *measure* on a topological space, we always mean a probability Borel measure. Let W be an arbitrary compact space and let ν_n $(n = 0, 1, \cdots)$ and ν be measures on W. We say that ν_n *converges weakly* [11] to ν as $n \to \infty$, and denote $w-\lim_{n\to\infty} \nu_n = \nu$, if for any real-valued continuous function f on W, $\int f d\nu_n$ converges to $\int f d\nu$ as $n \to \infty$. For $x \in W$, δ_x is the unit measure at x. By a *non-degenerate* measure on W, we mean a measure which is not a unit measure.

Let $\alpha \in \Sigma^N$. Let Ξ_α denote the family of all infinite subsets S of N such that

$$\mu_\alpha^S = w - \lim_{\substack{n \in S \\ n \to \infty}} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i\alpha} \tag{1.1}$$

exists. Note that $\Xi_\alpha \neq \phi$ for any $\alpha \in \Sigma^N$ since the space of measures on a compact metric space is compact in the topology of weak convergence (see [11]). Also, note

Received February 1, 1973 and in revised form April 10, 1973

that μ_α^S is a T-invariant measure for any $\alpha \in \Sigma^N$ and $S \in \Xi_\alpha$. We call $\alpha \in \Sigma^N$ a *stochastic sequence* [3] (or sometimes a *quasi-regular point* in Σ^N with respect to T [9]) if $N \in \Xi_\alpha$. In this case, we denote $\mu_\alpha = \mu_\alpha^N$. Let P be a non-degenerate measure on Σ. A stochastic sequence $\alpha \in \Sigma^N$ is called a *P-normal sequence* if μ_α equals P^N, the product measure of P on Σ^N. Note that if $\Sigma = \{0, 1, \cdots, r-1\}$ and $P(\{i\}) = 1/r$ for $i = 0, 1, \cdots, r-1$, then the notion of P-normal sequence coincides with the usual notion of r-adic normal sequence. The set of all P-normal sequences is denoted by Nor_P. A strictly increasing function from N to N is called a *selection function*. Let τ be a selection function. For $\alpha \in \Sigma^N$, the subsequence of α selected by τ is defined by $(\alpha \circ \tau)(i) = \alpha(\tau(i))$ for any $i \in N$ and denoted by $\alpha \circ \tau$. Following von Mises, $\alpha \in \Sigma^N$ is called a *τ-collective* if

$$(1.2) \qquad w - \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\alpha(i)} = w - \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\alpha(\tau(i))}.$$

Our problem is to characterize a selection function τ which satisfies the following conditions. Denote $\mathrm{Nor}_P \circ \tau = \{\alpha \circ \tau; \alpha \in \mathrm{Nor}_P\}$.

CONDITION 1. *Any $\alpha \in \mathrm{Nor}_P$ is a τ-collective.*

CONDITION 2. $\mathrm{Nor}_P \circ \tau \subset \mathrm{Nor}_P$.

CONDITION 3. $\mathrm{Nor}_P \circ \tau = \mathrm{Nor}_P$.

Clearly, Condition 3 implies Condition 2. It is also easy to verify that Condition 2 implies Condition 1.

In this paper, we prove that *each of the above three conditions is equivalent to Condition* 4 *stated below under a reasonable restriction that* $\overline{\lim}_{n \to \infty} \tau(n)/n < \infty$ (Theorem 4). It should be remarked that the fact that Condition 4 implies Condition 2 under the restriction stated above was already obtained by Weiss [14] in 1971.

To state Condition 4, some more notions are necessary. For a selection function τ, denote by $\theta_\tau \in \{0,1\}^N$ the 0-1 sequence defined by

$$(1.3) \qquad \theta_\tau(i) = \begin{cases} 1 & \text{if } i \in \{\tau(j); j \in N\} \\ 0 & \text{else} \end{cases} \qquad (i \in N).$$

That is, $\theta_\tau(i) = 1$ if and only if the i-th coordinate is selected as a subsequence by the selection function τ. For a T-invariant measure μ on $\{0,1\}^N$, where T is the shift on $\{0,1\}^N$, the *entropy* of the measure-preserving transformation T on the measure space $(\{0,1\}^N, \mu)$ is denoted by $h_\mu(T)$. That is,

$$(1.4) \qquad h_\mu(T) = \lim_{n\to\infty} \frac{1}{n} \sum_{\xi\in\{0,1\}^n} -\mu(\Gamma_\xi)\cdot\log\mu(\Gamma_\xi),$$

where for $\xi = (\xi_0, \xi_1, \cdots, \xi_{n-1}) \in \{0,1\}^n$,

$$(1.5) \qquad \Gamma_\xi = \{\beta\in\{0,1\}^N; \beta(i) = \xi_i \text{ for } i = 0,1,\cdots,n-1\}.$$

The above limit is known to exist [2]. Following [14], $\beta \in \{0,1\}^N$ is said to be *completely deterministic* if $h_\mu(T) = 0$ for any $\mu \in \{\mu_\beta^S; S \in \Xi_\beta\}$. Now, we state Condition 4.

CONDITION 4. *θ_τ is completely deterministic.*

Note that Condition 4 is indifferent to what Σ and P are. This condition is not only simple but also easy to check. Various types of sequences are known to be completely deterministic (Example 1). It seems that the notion of completely deterministic 0-1 sequence has some significance from the constructive point of view. In fact, we prove some closure properties of the class of completely deterministic 0-1 sequences under operations to construct a new sequence from other sequences (§5).

We also remark that Theorem 2 is a sort of extension theorem concerning invariant measures and has some applications to ergodic theory, which will be discussed elsewhere.

Let us explain some notation used in this paper. By Σ, Σ_1 and Σ_2, we always represent compact metric spaces. Note that the product space Σ^N is also a compact metric space. As a topology on a finite set, we always consider the discrete topology. We use the same notation T for shifts on different spaces. The shift on a product space $\Sigma_1^N \times \Sigma_2^N$ is defined by $T(\alpha,\beta) = (T\alpha, T\beta)$, where $\alpha\in\Sigma_1^N$ and $\beta\in\Sigma_2^N$. Let W be any compact space. By $C(W)$, we mean the Banach space consisting of all real-valued continuous functions on W with the norm $\|f\| = \sup_{x\in W}|f(x)|$. Let $q < p$ be non-negative integers. We consider $C(\Sigma^q)$ as a subspace of $C(\Sigma^p)$ and $C(\Sigma^N)$ identifying $f\in C(\Sigma^q)$ with the functions $(\xi_0,\cdots,\xi_{p-1}) \to f(\xi_0,\cdots,\xi_{q-1})$ and $\alpha \to f(\alpha(0),\cdots,\alpha(q-1))$, where $(\xi_0,\cdots,\xi_{p-1})\in\Sigma^p$ and $\alpha\in\Sigma^N$, respectively. Also, $C(\Sigma_1^N)$ and $C(\Sigma_2^N)$ are considered as subspaces of $C(\Sigma_1^N\times\Sigma_2^N)$ in the same manner. Let ν_i $(i = 1,2)$ be a measure on Σ_i^N. By $\nu_1\times\nu_2$, we mean the product measure of ν_1 and ν_2 on $\Sigma_1^N\times\Sigma_2^N$. For a measure ν on $\Sigma_1^N\times\Sigma_2^N$, the marginal distribution on Σ_i^N $(i = 1,2)$ is denoted by ν/Σ_i^N.

For a subset E of a set X, E^c denotes the complement of E in X. For a finite set E, $|E|$ denotes the number of elements in E. For $n\in N$, denote

$$N_n = \{0, 1, \cdots, n-1\}.$$

For an infinite subset S of N and a subset E of N, we denote

$$(1.6)\qquad \begin{cases} \underline{\sigma}_S(E) = \varliminf\limits_{\substack{n \in S \\ n \to \infty}} \dfrac{1}{n} |N_n \cap E| \\ \sigma_S(E) = \lim\limits_{\substack{n \in S \\ n \to \infty}} \dfrac{1}{n} |N_n \cap E| \\ \sigma(E) = \sigma_N(E). \end{cases}$$

In the rest of this section, we list up fundamental properties about weak convergence of measures which will be often used in the following. See [11] for the proofs. Let W be any compact metric space. Then, $C(W)$ has a countable base as a topological vector space. Let ν_n $(n \in N)$ and ν be measures on W. Then, $w-\lim_{n\to\infty} \nu_n = \nu$ if and only if $\lim_{n\to\infty} \int f d\nu_n = \int f d\nu$ for any f belonging to a base of W. Assume that $w-\lim_{n\to\infty} \nu_n = \nu$. Then, for any open set D of W, $\varliminf_{n\to\infty} \nu_n(D) \geqq \nu(D)$. Also, for any bounded real-valued function f on W which is continuous at almost all points with respect to ν, it holds that $\lim_{n\to\infty} \int f d\nu_n = \int f d\nu$. Let ψ be a mapping from W to another compact metric space W' which is continuous at almost all points with respect to ν. Then, $\nu \circ \psi^{-1} = w-\lim_{n\to\infty} (\nu_n \circ \psi^{-1})$, where $\nu \circ \psi^{-1}$'s are the induced measures on W'. Particularly, $\mu^S_{(\alpha_1,\alpha_2)}/\Sigma_i^N = \mu^S_{\alpha_i}$ for $i = 1,2$, where $(\alpha_1, \alpha_2) \in \Sigma_1^N \times \Sigma_2^N$ and $S \in \Xi_{(\alpha_1,\alpha_2)}$. Let μ be a T-invariant measure on Σ^N. Then,

$$(1.7)\qquad \mu(\{\alpha \in \Sigma^N;\ \mu_\alpha = \mu\}) = 1 \text{ or } 0$$

according to whether μ is ergodic with respect to T or not [9].

2. Sequences and invariant measures

LEMMA 2.1. *Let $\alpha_i \in \Sigma^N$ $(i = 1,2)$ be stochastic sequences such that $\mu_{\alpha_1} = \mu_{\alpha_2}$. Let D be a subset of N such that*

$$(2.1)\qquad \sigma(\{i;\ i \in D,\ i+1 \in D^c\}) = 0.$$

Let $D = \{d_0 < d_1 < \cdots\}$ and $D^c = \{d'_0 < d'_1 < \cdots\}$. Define $\beta \in \Sigma^N$ by

$$(2.2)\qquad \beta(i) = \begin{cases} \alpha_1(j) & \text{if } i = d_j \\ \alpha_2(j) & \text{if } i = d'_j \end{cases} \qquad (i \in N).$$

Then, β is a stochastic sequence such that $\mu_\beta = \mu_{\alpha_i}$ $(i = 1,2)$.

PROOF. It is sufficient to prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i\beta) = \int f d\mu_{\alpha_1} \tag{2.3}$$

for any $q \in N$ and $f \in C(\Sigma^q)$, since $\bigcup_{q=0}^{\infty} C(\Sigma^q)$ is dense in $C(\Sigma^N)$. Let $f \in C(\Sigma^q)$. Let

$$E = \{i \in N; \{i, i+1, \cdots, i+q-1\} \text{ is contained either in } D \text{ or in } D^c\}. \tag{2.4}$$

It follows from (2.1) that $\sigma(E) = 1$. For $i \in N$, denote

$$u_i = \begin{cases} 1 & \text{if } i \in D \\ 2 & \text{if } i \in D^c, \end{cases}$$

$$v_1(i) = |D \cap N_i|, \text{ and}$$

$$v_2(i) = |D^c \cap N_i|.$$

Then, from (2.2) and (2.4), for any $i \in E$, we have

$$f(T\beta) = f(T^{v_{u_i}(i)}\alpha_{u_i}).$$

Therefore,

$$\begin{aligned}
\sum_{i=0}^{n-1} f(T^i\beta) &= \sum_{i=0}^{n-1} fT(^{v_{u_i}(i)}\alpha_{u_i}) + o(n) \\
&= \sum_{i=0}^{v_1(n)} f(T^i\alpha_1) + \sum_{i=0}^{v_2(n)} f(T^i\alpha_2) + o(n) \\
&= v_1(n)\int f d\mu_{\alpha_1} + o(v_1(n)) + v_2(n)\int f d\mu_{\alpha_2} + o(v_2(n)) + o(n) \\
&= n\int f d\mu_{\alpha_1} + o(n).
\end{aligned}$$

This completes the proof of (2.3).

THEOREM 1. *Let $\alpha \in \Sigma^N$ and $S \in \Xi_\alpha$, assume that μ_α^S is ergodic with respect to T. Then, there exists a stochastic sequence $\beta \in \Sigma^N$ such that*

$$\sigma_S(\{i; \alpha(i) = \beta(i)\}) = 1$$

(and hence $\mu_\alpha^S = \mu_\beta$).

The idea of the proof. Let ν be any ergodic measure on Σ^N. Then, for any $n \in N$, any open neighbourhood W (in the topology of the weak convergence) of ν/Σ^n and any $\varepsilon > 0$, there exists $k \in N$ such that

$$\nu/\Sigma^k(\{(\xi_0, \cdots, \xi_{k-1}) \in \Sigma^k; \frac{1}{k-n+1} \sum_{i=0}^{k-n} \delta_{(\xi_i, \cdots, \xi_{i+n-1})} \in W\}) > 1 - \varepsilon.$$

Let $\nu = \mu_\alpha^S$. Then, this implies that

$$\underline{\sigma}_S\left(\left\{j \in N; \frac{1}{k-n+1} \sum_{i=j}^{j+k-n} \delta_{(\alpha(i), \cdots, \alpha(i+n-1))} \in W\right\}\right) > 1 - \varepsilon.$$

Collecting such sections $\{j,\cdots,j+k-1\}$ of integers that satisfy

$$\frac{1}{k-n+1}\sum_{i=j}^{j+k-n}\delta_{(\alpha(i),\cdots,\alpha(i+n-1))}\in W$$

for sufficiently large n's and sufficiently small W's, we can construct a subset $D=\{d_0<d_1<\cdots\}$ of N such that $\sigma_S(D)=1$ and $\mu_{\alpha_1}=\nu$, where $\alpha_1(i)=\alpha(d_i)$. Combining this fact with Lemma 1, we can prove Theorem 1.

PROOF. Put $\nu=\mu_\alpha^S$. We take a countable base $F=\{f_1,f_2,\cdots\}$ of $C(\Sigma^N)$ such that

(2.5) $$\|f_j\|\leqq 1 \quad (j=1,2,\cdots), \text{ and}$$

(2.6) $$f_j\in C(\Sigma^j) \quad (j=1,2,\cdots).$$

Since ν is ergodic with respect to T, we have

(2.7) $$\lim_{n\to\infty}\nu\left(\left\{\gamma;\ \sup_{1\leqq j\leqq k}\left|\frac{1}{n}\sum_{i=0}^{n-1}f_j(T^i\gamma)-\int f_j d\nu\right|<\frac{1}{k}\right\}\right)=1$$

for any $k=1,2,\cdots$. Therefore, we can take $r_1<r_2<\cdots$ such that

(2.8) $$r_k>k^2 \quad (k=1,2,\cdots), \text{ and}$$

(2.9) $$\nu\left(\left\{\gamma;\ \sup_{1\leqq j\leqq k}\left|\frac{1}{r_k}\sum_{i=0}^{r_k-1}f_j(T^i\gamma)-\int f_j d\nu\right|<\frac{1}{k}\right\}\right)>1-\frac{1}{k} \quad (k=1,2,\cdots).$$

Put

(2.10) $$\Gamma_k=\left\{\gamma;\ \sup_{1\leqq j\leqq k}\left|\frac{1}{r_k}\sum_{i=0}^{r_k-1}f_j(T^i\gamma)-\int f_j d\nu\right|<\frac{1}{k}\right\}.$$

Since

$$\nu=w-\lim_{\substack{n\in S\\ n\to\infty}}\frac{1}{n}\sum_{i=0}^{n-1}\delta_{T^i\alpha}$$

and Γ_k is an open set, it follows from (2.9) that

(2.11) $$\underline{\sigma}_S(\{i;\ T^i\alpha\in\Gamma_k\})\geqq\nu(\Gamma_k)>1-\frac{1}{k}.$$

For $k=1,2,\cdots$ and $m=1,2,\cdots$, define a_m^k inductively by

(2.12) $$\begin{aligned} a_1^k &= \min\{i;\ T^i\alpha\in\Gamma_k\}, \text{ and}\\ a_{m+1}^k &= \min\{i;\ i\geqq a_m^k+r_k+k,\ T^i\alpha\in\Gamma_k\}.\end{aligned}$$

Put

(2.13) $$E_k=\bigcup_{m=1}^{\infty}\{a_m^k,a_m^k+1,\cdots,a_m^k+r_k+k-1\} \qquad (k=1,2,\cdots).$$

It is clear that if $i \notin E_k$, then $T^i\alpha \notin \Gamma_k$. Hence, from (2.11),

$$\underline{\sigma}_S(E_k) \geqq \underline{\sigma}_S(\{i;\ T^i\alpha \in \Gamma_k\}) > 1 - \frac{1}{k}. \tag{2.14}$$

Therefore, there exist $0 = h_1 < h_2 < \cdots$ such that

$$h_k \geqq k h_{k-1}, \tag{2.15}$$

$$|E_{k-1} \cap \{h_{k-1}, h_{k-1}+1, \cdots, h_k - 1\}| \geqq k^2 \cdot r_k, \tag{2.16}$$

$$h_k \in S \text{ for } k = 2, 3, \cdots, \text{ and} \tag{2.17}$$

$$\frac{1}{n}|N_n \cap E_k| \geqq 1 - \frac{2}{k} \text{ for any } n \in S \text{ with } n \geqq h_k. \tag{2.18}$$

Put

$$E = \bigcup_{k=1}^{\infty} (E_k \cap \{h_k, h_k + 1, \cdots, h_{k+1} - 1\}). \tag{2.19}$$

For $n \in N$, let $t(n)$ denote k such that $h_k \leqq n < h_{k+1}$. Let $n \in S$ and $k = t(n) \geqq 2$. Then,

$$\begin{aligned} &|E \cap N_n| \\ &\geqq |E_{k-1} \cap \{h_{k-1}, \cdots, h_k - 1\}| + |E_k \cap \{h_k, \cdots, n-1\}| \\ &\geqq |E_{k-1} \cap N_{h_k}| - h_{k-1} + |E_k \cap N_n| - h_k \\ &\geqq \left(1 - \frac{2}{k-1}\right)h_k - h_{k-1} + \left(1 - \frac{2}{k}\right)n - h_k \text{ (from (2.17) and (2.18))} \\ &\geqq n - \frac{4n}{k} - h_{k-1} \\ &\geqq n - \frac{5n}{k} \quad \text{(from (2.15))}. \end{aligned} \tag{2.20}$$

Hence, $\sigma_S(E) = 1$. Define E' and E'' by

$$\begin{aligned} E' &= \{n \in E;\ n = a_m^{t(n)} \text{ for some } m\}, \text{ and} \\ E'' &= \{n \in E';\ n + r_{t(n)} + t(n) \leqq h_{t(n)+1}\}. \end{aligned} \tag{2.21}$$

Put

$$D = \bigcup_{n \in E''} \{n, n+1, \cdots, n + r_{t(n)} + t(n) - 1\}. \tag{2.22}$$

Then, it is clear that

$$\sigma(\{i;\ i \in D,\ i+1 \in D^c\}) = 0 \tag{2.23}$$

since $\lim_{n\to\infty} t(n) = \infty$. It is also clear from (2.13), (2.19), (2.21) and (2.22) that $D \subset E$ and

(2.24) $E \cap D^c \subset \{n \in E;\ n - r_{t(n)} - t(n) < h_{t(n)} \text{ or } n + r_{t(n)} + t(n) > h_{t(n)+1}\}$.

Let $n \in N$ and $k = t(n) \geqq 2$. Then,

$$\begin{aligned}\frac{|D \cap N_n|}{|E \cap N_n|} &\geqq 1 - \frac{\sum_{i=1}^{k} 2(r_i + i)}{|E \cap N_n|} && \text{(from (2.24))}\\ &\geqq 1 - \frac{4kr_k}{|E_{k-1} \cap \{h_{k-1}, \cdots, h_k - 1\}|} && \text{(from (2.8))}\\ &\geqq 1 - \frac{4}{k} && \text{(from (2.16)).}\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \frac{|D \cap N_n|}{|E \cap N_n|} = 1. \tag{2.25}$$

This implies that $\sigma_S(D) = 1$ since $\sigma_S(E) = 1$. Let $D = \{d_0 < d_1 < \cdots\}$ and $E'' = \{e_0 < e_1 < \cdots\}$. For $m = 0, 1, \cdots$, put $c_m = |D \cap N_{e_m}|$. Then, it follows from (2.22) that if $0 \leqq j < r_{t(m)} + t(m)$, then

$$c_{m+1} = c_m + r_{t(e_m)} + t(e_m) \text{ and } d_{c_m + j} = e_m + j. \tag{2.26}$$

Define $\alpha_1 \in \Sigma^N$ by $\alpha_1(i) = \alpha(d_i)$ $(i \in N)$. Then, it is clear from (2.26) that

$$\alpha(e_m + j) = \alpha_1(c_m + j) \text{ if } 0 \leqq j < r_{t(e_m)} + t(e_m). \tag{2.27}$$

For $j \geqq 1$, assume that $t(e_m) \geqq j$. Let $k = t(e_m)$. Then,

$$\begin{aligned}&\left|\sum_{i=c_m}^{c_{m+1}-1} f_j(T^i\alpha_1) - \sum_{i=e_m}^{e_m+r_k-1} f_j(T^i\alpha)\right|\\ &= \left|\sum_{i=c_m+r_k}^{c_{m+1}-1} f_j(T^i\alpha_1)\right| \quad \text{(from (2.6) and (2.27))}\\ &\leqq k \quad \text{(from (2.5) and (2.26)).}\end{aligned} \tag{2.28}$$

On the other hand, since $T^{e_m}\alpha \in \Gamma_k$ from (2.12) and (2.21), it holds from (2.10) that

$$\left|\sum_{i=e_m}^{e_m+r_k-1} f_j(T^i\alpha) - r_k \cdot \int f_j d\nu\right| < \frac{r_k}{k}. \tag{2.29}$$

Combining (2.28) and (2.29), we have

$$\left|\sum_{i=c_m}^{c_{m+1}-1} f_j(T^i\alpha_1) - (c_{m+1} - c_m)\int f_j d\nu\right| \leqq 2k + \frac{r_k}{k}. \tag{2.30}$$

Here, we also used (2.5) and (2.26). Thus,

$$\left|\frac{1}{c_{m+1}-c_m}\sum_{i=c_m}^{c_{m+1}-1} f_j(T^i\alpha_1) - \int f_j dv\right| \leqq \frac{2k}{r_k} + \frac{1}{k} \leqq \frac{3}{k} \tag{2.31}$$

from (2.26), (2.30) and (2.8). Since

$$\frac{1}{c_n}\sum_{i=0}^{c_n-1} f_j(T^i\alpha_1) = \sum_{m=1}^{n-1} \frac{c_{m+1}-c_m}{c_n} \cdot \frac{1}{c_{m+1}-c_m}\sum_{i=c_m}^{c_{m+1}-1} f_j(T^i\alpha_1),$$

(2.31) leads us to the conclusion that

$$\lim_{n\to\infty} \frac{1}{c_n}\sum_{i=0}^{c_n-1} f_j(T^i\alpha_1) = \int f_j dv. \tag{2.32}$$

Note that j is arbitrary in (2.32). Moreover since

$$\begin{aligned}
\overline{\lim_{m\to\infty}} \frac{c_{m+1}-c_m}{c_m} &= \overline{\lim_{m\to\infty}} \frac{r_{t(e_m)}+t(e_m)}{|D\cap N_{e_m}|} \quad \text{(from (2.26))} \\
&\leqq \overline{\lim_{n\to\infty}} \frac{r_{t(n)}+t(n)}{|D\cap N_n|} \\
&= \overline{\lim_{n\to\infty}} \frac{r_{t(n)}+t(n)}{|E\cap N_n|} \quad \text{(from (2.25))} \\
&\leqq \overline{\lim_{n\to\infty}} \frac{2r_{t(n)}}{|E_{t(n)-1}\cap\{h_{t(n)-1},\cdots,h_{t(n)}-1\}|} \\
&\leqq \overline{\lim_{n\to\infty}} \frac{2}{t(n)^2} \quad \text{(from (2.16))} \\
&= 0,
\end{aligned}$$

it follows from (2.32) that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f_j(T^i\alpha_1) = \int f_j dv \qquad (j = 1,2,\cdots).$$

Thus, α_1 is a stochastic sequence such that $\mu_{\alpha_1} = v$. Let $\alpha_2 \in \Sigma^N$ be any stochastic sequence such that $\mu_{\alpha_2} = v$ (for example $\alpha_2 = \alpha_1$). Applying Lemma 1.1 for D, α_1 and α_2 (note (2.23)), we obtain a stochastic sequence $\beta \in \Sigma^N$ such that $\alpha(i) = \beta(i)$ if $i \in D$. This completes the proof since $\sigma_S(D) = 1$.

THEOREM 2. *Let v be a T-invariant measure on $\Sigma_1^N \times \Sigma_2^N$. Assume that $\beta \in \Sigma_2^N$ and $S \in \Xi_\beta$ satisfy that $\mu_\beta^S = v/\Sigma_2^N$. Then, there exists $\alpha \in \Sigma_1^N$ such that $S \in \Xi_{(\alpha,\beta)}$ and $\mu_{(\alpha,\beta)}^S = v$. Moreover, if v/Σ_1^N is ergodic with respect to T, then the above α can be taken as a stochastic sequence.*

Applying this theorem to a trivial case that $\beta = (x, x, \cdots)$, $S = N$ and $\nu = \mu \times \delta_\beta$, we have the following result which is known as Kakutani's theorem [10].

COROLLARY (S. Kakutani). *For any T-invariant measure μ on Σ^N, there exists a stochastic sequence $\alpha \in \Sigma^N$ such that $\mu_\alpha = \mu$.*

PROOF OF THEOREM 2. Let ν be a T-invariant measure on $\Sigma_1^N \times \Sigma_2^N$. Let $\beta \in \Sigma_2^N$ and $S \in \Xi_\beta$ satisfy that $\mu_\beta^S = \nu / \Sigma_2^N$. The latter statement of the theorem follows from the former half of the theorem and Theorem 1. In fact, if ν / Σ_1^N is ergodic with respect to T and if we can take $\alpha \in \Sigma_1^N$ such that $S \in \Xi_{(\alpha,\beta)}$ and $\mu_{(\alpha,\beta)}^S = \nu$, then from Theorem 1, we can take a *stochastic sequence* α' such that $\sigma_S(\{i; \alpha'(i) = \alpha(i)\}) = 1$. Then, it is clear that α' also satisfies that $S \in \Xi_{(\alpha',\beta)}$ and $\mu_{(\alpha',\beta)}^S = \nu$. Therefore, it is sufficient to prove the first half of the theorem.

Take a countable base $\{f_1, f_2, \cdots\}$ of $C(\Sigma_1^N \times \Sigma_2^N)$ such that

$$\|f_j\| \leqq 1 \qquad (j = 1, 2, \cdots), \text{ and} \tag{2.33}$$

$$f_j \in C(\Sigma_1^j \times \Sigma_2^j) \qquad (j = 1, 2, \cdots). \tag{2.34}$$

Take a sequence of real numbers $1 > \varepsilon_1 > \varepsilon_2 > \cdots$ which tends to 0. For each $h = 1, 2, \cdots$, we can prove that there exists $\alpha_h \in \Sigma_1^N$ such that

$$\sup_{1 \leqq j \leqq h} \overline{\lim_{\substack{n \in S \\ n \to \infty}}} \left| \frac{1}{n} \sum_{i=0}^{n-1} f_j(T^i \alpha_h, T^i \beta) - \int f_j d\nu \right| \leqq \varepsilon_h / 2. \tag{2.35}$$

We will defer the proof of this fact for a while.

Next, for each $h = 1, 2, \cdots$, select $t_h > 0$ such that

$$\sup_{1 \leqq j \leqq h} \overline{\sup_{\substack{n \in S \\ n \geqq t_h}}} \left| \frac{1}{n} \sum_{i=0}^{n-1} f_j(T^i \alpha_h, T^i \beta) - \int f_j d\nu \right| \leqq \varepsilon_h. \tag{2.36}$$

Let k_h satisfy that $k_1 = 0$,

$$k_h \in S \ (h = 2, 3, \cdots), \text{ and} \tag{2.37}$$

$$k_h \geqq \max \left\{ t_h, \frac{k_{h-1} + 2h}{\varepsilon_h} \right\} \qquad (h = 2, 3, \cdots). \tag{2.38}$$

Define $\alpha \in \Sigma_1^N$ as follows:

$$\alpha(i) = \alpha_h(i) \text{ for every } i \in \{k_h, \cdots, k_{h+1} - 1\} \qquad (h = 1, 2, \cdots). \tag{2.39}$$

For this α, we prove that $S \in \Xi_{(\alpha,\beta)}$ and $\mu_{(\alpha,\beta)}^S = \nu$. To prove this it is sufficient to prove that

$$\lim_{\substack{n\in S\\ n\to\infty}} \frac{1}{n}\sum_{i=0}^{n-1} f_j(T^i\alpha, T^i\beta) = \int f_j dv \qquad (j=1,2,\cdots). \tag{2.40}$$

Let $j \geqq 1$ and $\varepsilon > 0$ be arbitrary. Select m such that $m \geqq j$ and $\varepsilon_m \leqq \varepsilon/7$. Consider an arbitrary n satisfying that $n \in S$ and $n \geqq k_{m+1}$. Select h such that $k_{h+1} \leqq n < k_{h+2}$. Clearly, $j \leqq m \leqq h$. Put $I_1 = \{i \in N;\, k_{h+1} \leqq i \leqq n-j\}$ and $I_0 = \{i \in N;\, k_h \leqq i \leqq k_{h+1} - j\}$. Then, it is clear from (2.34) and (2.39) that

$$\begin{aligned} &f_j(T^i\alpha, T^i\beta) = f_j(T^i\alpha_{h+1}, T^i\beta) \text{ if } i \in I_1, \text{ and}\\ &f_j(T^i\alpha, T^i\beta) = f_j(T^i\alpha_h, T^i\beta) \text{ if } i \in I_0. \end{aligned} \tag{2.41}$$

Therefore, from (2.33), (2.38) and (2.41), it holds that

$$\sum_{i=0}^{n-1} f_j(T^i\alpha, T^i\beta) = \sum_{i\in I_1} f_j(T^i\alpha_{h+1}, T^i\beta) + \sum_{i\in I_0} f_j(T^i\alpha_h, T^i\beta) + C_1, \tag{2.42}$$

where $|C_1| \leqq k_h + 2j \leqq k_h + 2(h+1) \leqq k_{h+1}\cdot\varepsilon_{h+1} \leqq n\varepsilon_h$. In addition, from (2.33), (2.36), (2.37) and (2.38), we have

$$\begin{aligned} &\sum_{i\in I_1} f_j(T^i\alpha_{h+1}, T^i\beta)\\ &\quad = \sum_{i=0}^{n-1} f_j(T^i\alpha_{h+1}, T^i\beta) - \sum_{i=0}^{k_{h+1}-1} f_j(T^i\alpha_{h+1}, T^i\beta) - C_2\\ &\quad = n\left(\int f_j dv + C_3\right) - k_{h+1}\cdot\left(\int f_j dv + C_4\right) - C_2, \end{aligned} \tag{2.43}$$

where $|C_2| \leqq j \leqq h \leqq k_h\cdot\varepsilon_h \leqq n\varepsilon_h$, $|C_3| \leqq \varepsilon_{h+1} \leqq \varepsilon_h$ and $|C_4| \leqq \varepsilon_{h+1} \leqq \varepsilon_h$. Also

$$\begin{aligned} &\sum_{i\in I_0} f_j(T^i\alpha_h, T^i\beta)\\ &\quad = \sum_{i=0}^{k_{h+1}-1} f_j(T^i\alpha_h, T^i\beta) - \sum_{i=0}^{k_h-1} f_j(T^i\alpha_h, T^i\beta) - C_5\\ &\quad = k_{h+1}\cdot\left(\int f_j dv + C_6\right) - C_7 - C_5, \end{aligned} \tag{2.44}$$

where $|C_5| \leqq j \leqq n\varepsilon_h$, $|C_6| \leqq \varepsilon_h$ and $|C_7| \leqq k_h \leqq k_{h+1}\cdot\varepsilon_{h+1} \leqq n\varepsilon_h$. Combining (2.42), (2.43) and (2.44), we have

$$\left|\frac{1}{n}\sum_{i=0}^{n-1} f_j(T^i\alpha, T^i\beta) - \int f_j dv\right| \leqq 7\varepsilon_h \leqq \varepsilon,$$

which completes the proof of (2.40).

Now, it remains to prove (2.35). We have to show that

for any $\varepsilon > 0$, $q \in N$ *and a finite subset* F *of* $C(\Sigma_1^q \times \Sigma_2^q)$, *there exists* $\alpha \in \Sigma_1^N$ *such that*

$$\sup_{f\in F}\ \overline{\lim_{\substack{n\in S\\ n\to\infty}}}\ \left|\frac{1}{n}\sum_{i=0}^{n-1} f(T^i\alpha, T^i\beta) - \int f\,d\nu\right| \leqq \varepsilon. \tag{2.45}$$

The idea of the proof. We define a random mechanism by which a desired sequence is obtained with probability 1. Let L and R be sufficiently large integers. Let $M = LR$. Select an integer from $\{0, 1, \cdots, L-1\}$ with the equal probability $1/L$. Let it be t_0. For $k = 0, 1, \cdots, R-2$, select $\alpha(t_0 + kL)$, $\alpha(t_0 + kL + 1), \cdots$, $\alpha(t_0 + kL + L - 1)$ according to the probability

$$\nu\{\cdot \mid \beta(t_0 + kL), \cdots, \beta(t_0 + kL + L - 1)\}.$$

Select an integer t_1 again from $\{0, 1, \cdots, L-1\}$ with the equal probability $1/L$. For $k = 0, 1, \cdots, R-2$, select $\alpha(M + t_1 + kL), \cdots, \alpha(M + t_1 + kL + L - 1)$ according to the probability $\nu\{\cdot \mid \beta(M + t_1 + kL), \cdots, \beta(M + t_1 + kL + L - 1)\}$. Here, each of these random choices should be done independently of the others. Succeeding this procedure, we can define $\alpha(i)$ for most i's. Define $\alpha(i)$ arbitrary if it is not defined by this procedure.

PROOF OF (2.45).

Let $\varepsilon > 0$ and $q \in N$ be arbitrary. Without loss of generality, assume that

$$\sup_{f\in F} \|f\| \leqq 1. \tag{2.46}$$

Denote by X_i and Y_i ($i \in N$) the projections from $\Sigma_1^N \times \Sigma_2^N$ to Σ_1 and Σ_2 defined by $X_i(\gamma, \delta) = \gamma(i)$ and $Y_i(\gamma, \delta) = \delta(i)$, respectively, where $\gamma \in \Sigma_1^N$ and $\delta \in \Sigma_2^N$. We consider X_i and Y_i ($i \in N$) random variables on the measure space $(\Sigma_1^N \times \Sigma_2^N, \nu)$. Take a finite partition $Q = \{q_j\}$ of Σ_2 (Q is also identified with the partition $Y_0^{-1}Q$ of Σ_2^N) such that

$$\mu_\beta^S\left(\bigcup_j\ q_j^b\right) = 0, \tag{2.47}$$

where q_j^b is the boundary of q_j, and that

$$\sup\left\{|f(\gamma,\delta) - f(\gamma,\delta')|;\ f \in F,\ \gamma \in \Sigma_1^N,\ \delta \text{ and } \delta' \text{ belong to the same element of } \bigvee_{i=0}^{q-1} T^{-i}Q\right\} \leqq \varepsilon/3, \tag{2.48}$$

where the symbol "$\bigvee$" means the least common refinement of partitions. For $x \in \Sigma_2$, we denote by x^* the element of Q which contains x. In this sense, Y_i^* ($i \in N$) is a Q-valued random variable. For $f \in F$, define

(2.49) $$f_* = E_\nu\left\{f \middle| \Delta \times \bigvee_{i=0}^{q-1} T^{-i}Q\right\},$$

where Δ denotes the point-wise partition of Σ_1^N and $E_\nu\{\ |\ \}$ is the conditional expectation under the measure ν. Then, it is clear from (2.48) that

(2.50) $$\sup_{(\gamma,\delta)\in\Sigma_1^N\times\Sigma_2^N} |f(\gamma,\delta) - f_*(\gamma,\delta)| \leqq \varepsilon/3, \text{ and}$$
$$\int f_* d\nu = \int f d\nu.$$

Select an integer L such that $L > q$ and $2(q-1)/L \leqq \varepsilon/3$. Select an integer R such that $R \geqq 18/\varepsilon$. Put $M = RL$. Take a sequence of Σ_1-valued random variables $Z_0, Z_1, \cdots$ which are defined on some probability space such that

(2.51) $$\boldsymbol{Z}_0, \boldsymbol{Z}_1, \cdots \text{ are independent,}$$

where $\boldsymbol{Z}_i = (Z_{iM}, Z_{iM+1}, \cdots, Z_{iM+M-1})$ $(i = 0, 1, \cdots)$, and that

(2.52) $$\begin{aligned} &P_r\{Z_{iM} \in A_0, \cdots, Z_{iM+M-1} \in A_{M-1}\} \\ &= \frac{1}{L}\sum_{t=0}^{L-1}\Bigg(\prod_{k=0}^{t-1} \lambda(A_k) \cdot \prod_{k=M-L+t}^{M-1} \lambda(A_k) \times \\ &\quad \times \prod_{k=0}^{R-2} P_r\{X_0 \in A_{t+kL}, \cdots, X_{L-1} \in A_{t+kL+L-1} \,| \\ &\quad Y_0^* = \beta(iM + t + kL)^*, \cdots, Y_{L-1}^* = \beta(iM + t + kL + L - 1)^*\}\Bigg), \end{aligned}$$

where λ is any fixed measure on Σ_1 and $P_r\{\ |\ \}$ is the conditional probability about random variables. If $i \in N$ satisfies that

(2.53) $$L - 1 \leqq i - \left[\frac{i}{M}\right]M \leqq M - L - q,$$

where $[x]$ denotes the largest integer not greater than x, then it follows from (2.52) that for any measurable set A of Σ_1^q,

(2.54) $$\frac{1}{L}\sum_{t=0}^{L-q} m_{i,t}(A) \leqq P_r\{(Z_i, \cdots, Z_{i+q-1}) \in A\} \leqq \frac{1}{L}\sum_{t=0}^{L-q} m_{i,t}(A) + \frac{q-1}{L},$$

where

(2.55) $$m_{i,t}(A) = P_r\{(X_t, \cdots, X_{t+q-1}) \in A \,|\, Y_0^* = \beta(i-t)^*, \cdots, Y_{L-1}^* = \beta(i-t+L-1)^*\}.$$

Hence, from (2.46), it holds that, for any $f \in F$ and $i \in N$ satisfying (2.53),

$$\begin{aligned}(2.56)\qquad &\Big| E\{f_*(Z_i,\cdots,Z_{i+q-1},\beta(i)^*,\cdots,\beta(i+q-1)^*)\} - \\ &- \frac{1}{L-q+1}\sum_{t=0}^{L-q}\int f_*(x_0,\cdots,x_{q-1},\beta(i)^*,\cdots,\beta(i+q-1)^*)\times \\ &\times\ m_{i,t}(dx_0\cdots dx_{q-1})\Big| \leqq \varepsilon/3.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\int f_*(x_0,\cdots,x_{q-1},\beta(i)^*,\cdots,\beta(i+q-1)^*)m_{i,t}(dx_0\cdots dx_{q-1}) \\ &= \iint f_*(x_0,\cdots,x_{q-1},y_0^*,\cdots,y_{q-1}^*)\times \\ &\times\ m_{i,t}(dx_0\cdots dx_{q-1})\delta_{(\beta(i)^*,\dots,\beta(i+q-1)^*)}(dy_0^*\cdots dy_{q-1}^*) \\ (2.57)\qquad &= E_\nu\{f_*(X_t,\cdots,X_{t+q-1},Y_t^*,\cdots,Y_{t+q-1}^*)\,| \\ &\qquad Y_0^*=\beta(i-t)^*,\cdots,Y_{L-1}^*=\beta(i-t+L-1)^*\} \\ &= E_\nu\{f_*\circ T^t\,|\,Y_0^*,\cdots,Y_{L-1}^*\}(T^{i-t}\beta).\end{aligned}$$

Since

$$\sigma\left(\left\{i\in N;\, L-1\leqq i-\left[\frac{i}{M}\right]M\leqq M-L-q\right\}\right)=\frac{M-2L-q+2}{M}\geqq 1-\frac{\varepsilon}{6},$$

it follows from (2.56) and (2.57) that

$$\begin{aligned}&\overline{\lim_{n\to\infty}}\ \Big|\frac{1}{n}\sum_{i=0}^{n-1}E\{f_*(Z_i,\cdots,Z_{i+q-1},\beta(i)^*,\cdots,\beta(i+q-1)^*)\} \\ (2.58)\qquad &-\frac{1}{L-q+1}\sum_{t=0}^{L-q}\frac{1}{n}\sum_{i=t}^{n-1}E_\nu\{f_*\circ T^t\,|\,Y_0^*,\cdots,Y_{L-1}^*\}(T^{i-t}\beta)\Big|\leqq\frac{2\varepsilon}{3}.\end{aligned}$$

Since the function $E_\nu\{f_*\circ T^t\,|\,Y_0^*,\cdots,Y_{L-1}^*\}$ is continuous at any point except for the points belonging to the set $\bigcup_{i=0}^{L-1}\bigcup_j T^{-i}q_j^b$, and this set has measure 0 under μ_β^S from (2.47), it holds that

$$\begin{aligned}&\lim_{\substack{n\in S\\ n\to\infty}}\frac{1}{n}\sum_{i=t}^{n+t-1}E_\nu\{f_*\circ T^t\,|\,Y_0^*,\cdots,Y_{L-1}^*\}(T^{i-t}\beta) \\ (2.59)\qquad &=\int E_\nu\{f_*\circ T^t\,|\,Y_0^*,\cdots,Y_{L-1}^*\}d\mu_\beta^S=\int f_*\circ T^t d\nu=\int f_* d\nu=\int f d\nu.\end{aligned}$$

Hence, from (2.58), we have

$$\overline{\lim_{\substack{n\in S\\ n\to\infty}}} \left|\frac{1}{n}\sum_{i=0}^{n-1} E\{f_*(Z_i,\cdots,Z_{i+q-1},\beta(i)^*,\cdots,\beta(i+q-1)^*)\} - \right.$$

$$(2.60)\qquad \left. -\int f\,d\nu \right| \leqq \frac{2\varepsilon}{3}.$$

For $i\in N$ and $f\in F$, put $U_i^f = f(T^iZ, T^i\beta)$, where $Z=(Z_0,Z_1,\cdots)$. Then, it follows from (2.50) that

$$(2.61)\qquad |E\{U_i^f\} - E\{f_*(Z_i,\cdots,Z_{i+q-1},\beta(i)^*,\cdots,\beta(i+q-1)^*)\}| \leqq \frac{\varepsilon}{3}.$$

Combining (2.60) and (2.61), we have

$$(2.62)\qquad \overline{\lim_{\substack{n\in S\\ n\to\infty}}} \left|\frac{1}{n}\sum_{i=0}^{n-1} E\{U_i^f\} - \int f\,d\nu\right| \leqq \varepsilon.$$

Put $U_i^f = (U_{iM}^f, U_{iM+1}^f,\cdots,U_{iM+M-1}^f)$ $(i=0,1,\cdots)$. Then, it follows from (2.51) that $U_0^f, U_2^f,\cdots$ are independent, and $U_1^f, U_3^f,\cdots$ are independent. Hence the strong law of large number holds for $U_0^f, U_1^f,\cdots$;

$$P_r\left\{\lim_{n\to\infty} \left|\frac{1}{n}\sum_{i=0}^{n-1} U_i^f - \frac{1}{n}\sum_{i=0}^{n-1} E\{U_i^f\}\right| = 0\right\} = 1.$$

Since F is a finite set, this implies that

$$P_r\left\{\lim_{n\to\infty} \left|\frac{1}{n}\sum_{i=0}^{n-1} f(T^iZ,T^i\beta) - \frac{1}{n}\sum_{i=0}^{n-1} E\{U_i^f\}\right| = 0 \text{ for any } f\in F\right\} = 1.$$

In particular, there exists $\alpha\in\Sigma_1^N$ such that

$$\lim_{n\to\infty} \left|\frac{1}{n}\sum_{i=0}^{n-1} f(T^i\alpha,T^i\beta) - \frac{1}{n}\sum_{i=0}^{n-1} E\{U_i^f\}\right| = 0$$

for any $f\in F$. Combining this with (2.62), we have

$$\sup_{f\in F}\ \overline{\lim_{\substack{n\in S\\ n\to\infty}}} \left|\frac{1}{n}\sum_{i=0}^{n-1} f(T^i\alpha,T^i\beta) - \int f\,d\nu\right| \leqq \varepsilon,$$

which completes the proof of (2.45).

3. Two lemmas concerning disjointness

By an *endomorphism*, we mean a measure-preserving transformation on an abstract Lebesgue space. Let U be an endomorphism on (Ω,μ). By $h_\mu(U)$, we mean the *entropy* of U (see [2]). In this section, we shall discuss those results, which will be used in the next section, concerning the notion of *disjointness* between endomorphisms due to Furstenberg [3]. It should be remarked that Lemma 3.1 is proved following the idea used by Furstenberg in the proof of Theorem 1.1 in [3]. However, his proof contains a gap, which is filled by virtue of our Lemma 3.1. For this reason, we write the proof of Lemma 3.1 in detail. Lemma 3.2 is a special case of the so-called Pinsker's theorem which asserts that two endomorphisms, one of which has a *completely positive entropy* and the other of which has entropy 0, are disjoint (see [3], [12] or [13]).

LEMMA 3.1. *Let P be a non-degenerate measure on a compact metric space Σ. Let μ be a T-invariant measure on $\{0,1\}^N$ such that $h_\mu(T)>0$. Then, there exists a T-invariant measure ν on $\Sigma^N \times \{0,1\}^N$ such that*

(i) $\nu/\Sigma^N = P^N$,

(ii) $\nu/\{0,1\}^N = \mu$, *and*

(iii) X_0 *and* Y_0 *are not stochastically independent under* ν, *where* X_0 *and* Y_0 *are projections from* $\Sigma^N \times \{0,1\}^N$ *to* Σ *and* $\{0,1\}$, *respectively, defined by* $X_0(\alpha,\beta)=\alpha(0)$ *and* $Y_0(\alpha,\beta)=\beta(0)$ $(\alpha\in\Sigma^N,\ \beta\in\{0,1\}^N)$. *Moreover, if P is the Lebesgue measure on $\Sigma=[0,1]$, then we may choose the measure ν so that a version of $E_\nu\{Y_0 \mid X_0=t\}$ is a non-decreasing function of t which is non-constant, precisely* $E_\nu\{Y_0 \mid X_0=0+\} < E_\nu\{Y_0 \mid X_0=1-\}$.

PROOF. Let us first consider the special case when $\Sigma=[0,1]$ and $P=\lambda$ (the Lebesgue measure). Let μ be a T-invariant measure on $\{0,1\}^N$ such that $h_\mu(T)>0$. Denote by M the set of all negative integers. Let μ' be the measure on $\{0,1\}^M$ such that

$$\begin{aligned}(3.1)\qquad &\mu'(\{\beta\in\{0,1\}^M;\ \beta(i)=\xi_i \text{ for } i=-n,-n+1,\cdots,-1\})\\ &= \mu(\{\beta\in\{0,1\}^N;\ \beta(n+i)=\xi_i \text{ for } i=-n,-n+1,\cdots,-1\})\end{aligned}$$

for any $n\geqq 1$ and $\xi_{-n},\xi_{-n+1},\cdots,\xi_{-1}\in\{0,1\}$. For each $i\in N$, define a projection U_i from $[0,1]^N\times\{0,1\}^M$ to $[0,1]$ by $U_i(\alpha,\beta)=\alpha(i)$. Also, for each $i\in M$, define a projection V_i from $[0,1]^N\times\{0,1\}^M$ to $\{0,1\}$ by $V_i(\alpha,\beta)=\beta(i)$. Put

$$(3.2)\qquad \tau = \lambda^N \times \mu'.$$

Define a mapping $f:\{0,1\}^M\to[0,1]$ by

$$(3.3)\qquad f(\beta)=E_{\mu'}\{V_{-1} \mid V_{-2}=\beta(-1),\ V_{-3}=\beta(-2),\cdots\}.$$

Since $h_\mu(T) > 0$, we have

$$\mu'\{0 < f < 1\} \ (= \mu'(\{\beta; 0 < f(\beta) < 1\})) > 0. \tag{3.4}$$

Define a mapping $V_0: [0,1]^N \times \{0,1\}^M \to \{0,1\}$ by

$$V_0(\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha(0) \leqq f(\beta) \\ 0 & \text{else}. \end{cases} \tag{3.5}$$

Since f and U_0 are stochastically independent under τ because of (3.2), it holds for any $t \in [0,1]$ that

$$\begin{aligned} E_\tau\{V_0 \mid U_0 = t\} &= P_r\{f \geqq t \mid U_0 = t\} \\ &= P_r\{f \geqq t\} \\ &= \mu'\{f \geqq t\}, \text{ a.e.} \end{aligned} \tag{3.6}$$

Thus, from (3.4) and (3.6), $E_\tau\{V_0 \mid U_0 = t\}$ is a non-decreasing function of t such that $E_\tau\{V_0 \mid U_0 = 0+\} < E_\tau\{V_0 \mid U_0 = 1-\}$.

Define a mapping $\phi_1: [0,1]^N \times \{0,1\}^M \to [0,1]^N$ by $(\phi_1(\alpha,\beta))(i) = \alpha(i+1)$, where $\alpha \in [0,1]^N$, $\beta \in \{0,1\}^M$ and $i \in N$. Also, define a mapping $\phi_2: [0,1]^N \times \{0,1\}^M \to \{0,1\}^M$ by

$$(\phi_2(\alpha,\beta))(i) = \begin{cases} V_0(\alpha,\beta) & \text{if } i = -1 \\ \beta(i+1) & \text{if } i \leqq -2. \end{cases}$$

We prove that $\phi = (\phi_1, \phi_2)$ is a measure-preserving transformation on the measure space $([0,1]^N \times \{0,1\}^M, \tau)$. It is clear that ϕ_1 and ϕ_2 are independent under τ. Also, it is clear that $\lambda^N \circ \phi_1^{-1} = \lambda^N$. Therefore, it is sufficient to prove that $\mu' \circ \phi_2^{-1} = \mu'$. That is to say, the distribution of $(\cdots, V_{-2}, V_{-1})$ coincides with the distribution of $(\cdots, V_{-2} \circ \phi_2, V_{-1} \circ \phi_2)$, where V_i and $V_i \circ \phi_2$ $(i \in M)$ are considered as random variables on the measure space $([0,1]^N \times \{0,1\}^M, \tau)$. Note that $V_i \circ \phi_2 = V_{i+1}$ for $i \leqq -1$. Therefore, $(\cdots, V_{-2} \circ \phi_2, V_{-1} \circ \phi_2) = (\cdots, V_{-1}, V_0)$. Since μ is T-invariant, the distribution of $(\cdots, V_{-3}, V_{-2})$ equals the distribution of $(\cdots, V_{-2}, V_{-1})$. On the other hand,

$$\begin{aligned} &P_r\{V_0 = 1 \mid V_{-1} = \beta(-1), V_{-2} = \beta(-2), \cdots\} \\ &\quad = P_r\{f(\beta) \geqq U_0 \mid V_{-1} = \beta(-1), V_{-2} = \beta(-2), \cdots\} \\ &\quad = P_r\{t \geqq U_0\}\big|_{t = f(\beta)} \qquad (U_0 \perp\!\!\!\perp (\cdots, V_{-2}, V_{-1})) \\ &\quad = f(\beta) \\ &\quad = P_r\{V_{-1} = 1 \mid V_{-2} = \beta(-1), V_{-3} = \beta(-2), \cdots\} \qquad \text{(from (3.3))} \end{aligned}$$

for almost all (w.r.t. μ') $\beta \in \{0,1\}^M$. Thus, the distribution of $(\cdots, V_{-2}, V_{-1})$ equals the distribution of $(\cdots, V_{-2}\circ\phi_2, V_{-1}\circ\phi_2)$, and hence, ϕ preserves τ.

Define a mapping $\psi_1\colon [0,1]^N \times \{0,1\}^M \to [0,1]^N$ by $\psi_1(\alpha,\beta) = \alpha$. Define a mapping $\psi_2\colon [0,1]^N \times \{0,1\}^M \to \{0,1\}^N$ by $(\psi_2(\alpha,\beta))(i) = V_0(\phi^i(\alpha,\beta))$, $(i \in N)$. Let $\psi = (\psi_1, \psi_2)$. Then, it is clear that $\psi\circ\phi = T\circ\psi$, where T is the shift on $[0,1]^N \times \{0,1\}^N$. Define a measure ν on $[0,1]^N \times \{0,1\}^N$ by $\nu = \tau\circ\psi^{-1}$. Then, ν is T-invariant, since $\nu\circ T^{-1} = \tau\circ\psi^{-1}\circ T^{-1} = \tau\circ\phi^{-1}\circ\psi^{-1} = \tau\circ\psi^{-1} = \nu$. It is clear that $\nu/[0,1]^N = \lambda^N$. For each $i \in N$, let $Y_i\colon [0,1]^N \times \{0,1\}^N \to \{0,1\}$ be the projection such that $Y_i(\alpha,\beta) = \beta(i)$, Then, it is clear that for each $n \in N$, the distribution of $(Y_0, Y_1, \cdots, Y_{n-1})$ under ν equals the distribution of $(V_0, V_0\circ\phi, \cdots, V_0\circ\phi^{n-1})$ under τ. On the other hand, since $V_{-i}\circ\phi^i = V_0$ for any $i \in N$ and τ is ϕ-invariant, the latter coincides with the distribution of $(V_{-n}, V_{-n+1}, \cdots, V_{-1})$ under τ. Hence, $\nu/\{0,1\}^N = \mu$. Let $X_0\colon [0,1]^N \times \{0,1\}^N \to [0,1]$ be the projection such that $X_0(\alpha,\beta) = \alpha(0)$. Since the distribution of (X_0, Y_0) under ν equals the distribution of (U_0, V_0) under τ, $E_\nu\{Y_0 \mid X_0 = t\} = E_\tau\{V_0 \mid U_0 = t\}$ for any $t \in [0,1]$. Thus, $E_\nu\{Y_0 \mid X_0 = t\}$ is a non-decreasing function of t such that $E_\nu\{Y_0 \mid X_0 = 0+\} < E_\nu\{Y_0 \mid X_0 = 1-\}$. Note that this implies that

$$
\begin{aligned}
E_\nu\{Y_0 \mid 0 \leqq X_0 \leqq s\} &= \frac{1}{s}\int_0^s E_\nu\{Y_0 \mid X_0 = t\}\,dt \\
&< \frac{1}{1-s}\int_s^1 E_\nu\{Y_0 \mid X_0 = t\}\,dt = E_\nu\{Y_0 \mid s < X_0 \leqq 1\}
\end{aligned}
\tag{3.7}
$$

for any $0 < s < 1$.

Consider next the general case when Σ is a compact metric space and P is a non-degenerate measure on Σ. Since P is non-degenerate, there exists a measurable set S of Σ such that $0 < P(S) < 1$. There exists a measure-preserving mapping g from $([0,1], \lambda)$ to (Σ, P) such that $g^{-1}(S) = [0, P(S)]$. Define

$$\hat{g}\colon [0,1]^N \times \{0,1\}^N \to \Sigma^N \times \{0,1\}^N$$

by $\hat{g}(\alpha,\beta) = (\gamma,\beta)$, where $\gamma(i) = g(\alpha(i))$ for any $i \in N$. We prove that the measure $\nu\circ\hat{g}^{-1}$ on $\Sigma^N \times \{0,1\}^N$ satisfies the conditions of our lemma. Clearly, $\nu\circ\hat{g}^{-1}$ is T-invariant. It is also clear that $\nu\circ\hat{g}^{-1}/\Sigma^N = P^N$ and $\nu\circ\hat{g}^{-1}/\{0,1\}^N = \mu$. Let X'_0 and Y'_0 be projections from $\Sigma^N \times \{0,1\}^N$ to Σ and $\{0,1\}$, respectively, defined by $X'_0(\alpha,\beta) = \alpha(0)$ and $Y'_0(\alpha,\beta) = \beta(0)$. Then, from (3.7),

$$
\begin{aligned}
E_{\nu\circ\hat{g}^{-1}}\{Y'_0 \mid X'_0 \in S\} &= E_\nu\{Y_0 \mid 0 \leqq X_0 \leqq P(S)\} \\
&< E_\nu\{Y_0 \mid P(S) < X_0 \leqq 1\} = E_{\nu\circ\hat{g}^{-1}}\{Y'_0 \mid X'_0 \in S^c\}.
\end{aligned}
$$

Thus, X'_0 and Y'_0 are not stochastically independent under $\nu \circ \hat{g}^{-1}$, which completes the proof of Lemma 3.1.

The following is put here for later reference.

LEMMA 3.2 (see [3], [12] or [13]). *Let P be a measure on a compact metric space Σ. Let μ be a T-invariant measure on $\{0,1\}^N$ such that $h_\mu(T) = 0$. Then, for any T-invariant measure ν on $\Sigma^N \times \{0,1\}^N$ such that $\nu/\Sigma^N = P^N$ and $\nu/\{0,1\}^N = \mu$, it holds that $\nu = P^N \times \mu$.*

4. Subsequences of P-normal sequences

Let P be a non-degenerate measure on a compact metric space Σ. Let τ be a selection function. We here restate the four conditions introduced in §1.

CONDITION 1. *Any $\alpha \in \mathrm{Nor}_P$ is a τ-collective.*

CONDITION 2. $\mathrm{Nor}_P \circ \tau \subset \mathrm{Nor}_P$.

CONDITION 3. $\mathrm{Nor}_P \circ \tau = \mathrm{Nor}_P$.

CONDITION 4. *θ_τ is completely deterministic.*

Let X_0 be the projection $\Sigma^N \to \Sigma$ such that $X_0(\alpha) = \alpha(0)$ $(\alpha \in \Sigma^N)$. Let $\alpha \in \Sigma^N$ be a stochastic sequence. Then, it holds that

$$w - \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\alpha(i)} = \mu_\alpha \circ X_0^{-1}. \tag{4.1}$$

It then follows that Condition 2 implies Condition 1.

THEOREM 3. *Condition 1 implies Condition 4.*

PROOF. Assume that θ_τ is not completely deterministic. Then, there exists $S \in \Xi_{\theta_\tau}$ such that $h_{\mu^S_{\theta_\tau}}(T) > 0$. Let $\mu = \mu_{\theta_\tau}^S$. From Lemma 3.1, there exists a T-invariant measure ν on $\Sigma^N \times \{0,1\}^N$ such that

$$\nu/\Sigma^N = P^N, \tag{4.2}$$

$$\nu/\{0,1\}^N = \mu, \text{ and} \tag{4.3}$$

$$X_0 \text{ and } Y_0 \text{ are not stochastically independent under } \nu, \tag{4.4}$$

where X_0 and Y_0 are the projections from $\Sigma^N \times \{0,1\}^N$ to Σ and $\{0,1\}$, respectively, defined by $X_0(\gamma,\beta) = \gamma(0)$ and $Y_0(\gamma,\beta) = \beta(0)$. Note that $\nu/\Sigma^N = P^N$ is ergodic with respect to T. Applying Theorem 2 for these ν, θ_τ and S, we can select a *stochastic* sequence $\alpha \in \Sigma^N$ such that $S \in \Xi_{(\alpha,\theta\tau)}$ and $\mu^S_{(\alpha,\theta\tau)} = \nu$. Note that $\alpha \in \mathrm{Nor}_P$, since

$$\mu_\alpha = \mu_\alpha^S = \mu^S_{(\alpha,\theta_\cdot)}/\Sigma^N = P^N.$$

We prove that α is not a τ-collective. If

$$w - \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\alpha(\tau(i))}$$

does not exist, then α is not a τ-collective. Therefore, assume that this weak limit exists. It is then sufficient to prove that there exists $f \in C(\Sigma)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\alpha(i)) \neq \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\alpha(\tau(i))). \tag{4.5}$$

From (4.4), there exists $f \in C(\Sigma)$ such that

$$\frac{E_\nu\{f(X_0)Y_0\}}{E_\mu\{Y_0\}} \neq E_P\{f\}. \tag{4.6}$$

In this, note that $E_\mu\{Y_0\} \neq 0$. In fact, it follows from $h_\mu(T) > 0$ that $\mu\{Y_0 = 0\} \neq 1$. On the other hand, we have

$$E_P\{f\} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\alpha(i)), \text{ and} \tag{4.7}$$

$$\frac{E_\nu\{f(X_0)Y_0\}}{E_\mu\{Y_0\}} = \frac{\lim\limits_{\substack{n\in S\\ n\to\infty}} \frac{1}{n} \sum_{i=0}^{n-1} f(\alpha(i))\theta_\tau(i)}{\lim\limits_{\substack{n\in S\\ n\to\infty}} \frac{1}{n} \sum_{i=0}^{n-1} \theta_\tau(i)} = \lim_{\substack{n\in S\\ n\to\infty}} \frac{1}{t_n+1} \sum_{i=0}^{t_n} f(\alpha(\tau(i))), \tag{4.8}$$

where $t_n = \max\{i;\, \tau(i) \leqq n-1\}$ $(n = 1, 2, \cdots)$. Thus, (4.5) is proved by (4.6), (4.7) and (4.8). We complete the proof of Theorem 3.

Next, we prove that

(4.9) *Condition* 4 *implies Condition* 3 *under the hypothesis* $\overline{\lim}_{n\to\infty} \dfrac{\tau(n)}{n} < \infty$.

The conclusion of (4.9) (even the converse of Theorem 3) is not true in general unless the hypothesis $\overline{\lim}_{n\to\infty} \tau(n)/n < \infty$. For example, consider a selection function τ such that $\lim_{n\to\infty} \tau(n)/n = \infty$ is satisfied. Then, θ_τ is completely deterministic, since $\mu_{\theta_\cdot} = \delta_o$, where $\mathbf{0}(i) = 0$ for any $i \in N$. On the other hand, it is easily seen that $\mathrm{Nor}_P \circ \tau = \Sigma^N$; given $x \in \Sigma$, a P-normal sequence α such that

$(\alpha \circ \tau)\ (i) = x$ for any $i \in N$ is not a τ-collective. The proof of (4.9) follows after several lemmas. We first prove

LEMMA 4.1 ([14]). *Assume that*

$$\overline{\lim_{n\to\infty}}\ \frac{\tau(n)}{n} < \infty. \tag{4.10}$$

Then, Condition 4 *implies Condition* 2.

PROOF. Let us assume (4.10) and Condition 4. Let $\alpha \in \mathrm{Nor}_p$ be arbitrary. It is sufficient to prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(\alpha \circ \tau)) = \int f dP^N \tag{4.11}$$

for any $q \in N$ and $f \in C(\Sigma^q)$. Let $q \in N$ and $f \in C(\Sigma^q)$ be arbitrary. Let

$$\Delta = \left\{\beta \in \{0,1\}^N;\ \sum_{i=0}^{\infty}\ \beta(i) \geqq q\right\}.$$

Then, Δ is an open set. Define a mapping $\psi: \Sigma^N \times \Delta \to \Sigma^q$ by

$$(\psi(\gamma,\beta))(i) = \gamma\left(\min\left\{j;\ \sum_{k=0}^{j}\ \beta(k) = i+1\right\}\right) \qquad (i = 0,1,\cdots,q-1). \tag{4.12}$$

Then, ψ is a continuous mapping. Define a real-valued function g on $\Sigma^N \times \{0,1\}^N$ by

$$g(\gamma,\beta) = \begin{cases} f(\psi(\gamma,\beta)) & \text{if } \beta \in \Delta \text{ and } \beta(0) = 1 \\ 0 & \text{else.} \end{cases} \tag{4.13}$$

Clearly, g is continuous on a subset $\Sigma^N \times (\Delta \cup \{\mathbf{0}\})$, where $\mathbf{0}(i) = 0$ for any $i \in N$. Since $T^{-n}(\Sigma^N \times (\Delta \cup \{\mathbf{0}\}))$ $(n = 0,1,\cdots)$ is an increasing family of subsets such that the union equals the whole space $\Sigma^N \times \{0,1\}^N$, it holds that

$$\nu(\Sigma^N \times (\Delta \cup \{\mathbf{0}\})) = 1 \tag{4.14}$$

for any T-invariant measure ν on $\Sigma^N \times \{0,1\}^N$. To prove (4.11), it is sufficient to prove

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} g(T^i\alpha, T^i\theta_\tau)}{\sum_{i=0}^{n-1}\ \theta_\tau(i)} = \int f dP^N. \tag{4.15}$$

For any infinite subset S' of N, there exists a subset S of S' such that $S \in \Xi_{(\alpha,\theta_\tau)}$, since the space of measures is compact in the topology of weak convergence. Let

$\nu = \mu^S_{(\alpha,\theta_\cdot)}$ and $\mu = \mu^S_{\theta_\cdot}$. Since θ_τ is completely deterministic, $h_\mu(T) = 0$. Note that $\nu/\Sigma^N = P^N$ and $\nu/\{0,1\}^N = \mu$. Therefore, $\nu = P^N \times \mu$ from Lemma 3.2. Since g is continuous at almost all points of $\Sigma^N \times \{0,1\}^N$ with respect to ν, it holds that

$$\lim_{\substack{n\in S\\ n\to\infty}} \frac{\sum_{i=0}^{n-1} g(T^i\alpha, T^i\theta_\tau)}{\sum_{i=0}^{n-1} \theta_\tau(i)} = \frac{\lim\limits_{\substack{n\in S\\ n\to\infty}} \frac{1}{n}\sum_{i=0}^{n-1} g(T^i\alpha, T^i_{\theta_\cdot})}{\lim\limits_{\substack{n\in S\\ n\to\infty}} \frac{1}{n}\sum_{i=0}^{n-1} \theta_\tau(i)} = \frac{\int g d\nu}{\mu(\Gamma)}, \tag{4.16}$$

where $\Gamma = \{\beta; \beta(0) = 1\}$. In this, note that $\mu(\Gamma) > 0$ since (4.10). For $\beta \in \Gamma \cap \Delta$, it holds that

$$\begin{aligned} \int g(\gamma,\beta)P^N(d\gamma) &= \int f(\gamma(t_0), \gamma(t_1), \cdots, \gamma(t_{q-1}))P^N(d\gamma) \\ &= \int f dP^N, \end{aligned} \tag{4.17}$$

where $t_i = \min\{j; \sum_{k=0}^{j} \beta(k) = i+1\}$ $(i = 0,1,\cdots)$. If $\beta \notin \Gamma\cap\Delta$, then clearly $\int g(\gamma,\beta)\, P^N(d\gamma) = 0$. Therefore, from (4.17) and the fact that $\nu = P^N \times \mu$,

$$\begin{aligned} \int g d\nu &= \int_{\Gamma\cap\Delta} \left(\int g(\gamma,\beta)P^N(d\gamma)\right)\mu(d\beta) \\ &= \int_{\Gamma\cap\Delta} \left(\int f dP^N\right) d\mu \\ &= \mu(\Gamma\cap\Delta)\int f dP^N. \end{aligned} \tag{4.18}$$

Since $\mu(\Delta \cup \{\mathbf{0}\}) = 1$ and $\mathbf{0} \notin \Gamma$, we have

$$\mu(\Gamma\cap\Delta) = \mu(\Gamma). \tag{4.19}$$

It follows from (4.18) and (4.19) that (4.16) equals $\int f dP^N$. Since for any infinite subset S' of N, there exists an infinite subset S of S' as this, we complete the proof of (4.15). Thus, Lemma 4.1 is proved.

Let $\Gamma = \{\beta \in \{0,1\}^N; \beta(0) = 1\}$. Then, Γ is a closed set. For $\beta \in \Gamma$, let

$$t(\beta) = \begin{cases} \min\{i; i \geqq 1, \beta(i) = 1\} & \text{if } \beta(i) = 1 \text{ for some } i \geqq 1 \\ \infty & \text{else.} \end{cases} \tag{4.20}$$

Define a mapping $T_\Gamma: \Gamma \to \Gamma$ by

$$T_\Gamma \beta = \begin{cases} T^{t(\beta)}\beta & \text{if} \quad t(\beta) < \infty \\ \beta & \text{else.} \end{cases} \tag{4.21}$$

Let $W = \{1, 2, \cdots\} \cup \{\infty\}$ be the one-point compactification of the discrete space $\{1, 2, \cdots\}$. For $\beta \in \Gamma$, define $\psi(\beta) \in W^N$ by

$$\psi(\beta)(i) = t(T_\Gamma^i \beta) \qquad (i \in N). \tag{4.22}$$

Then, $\psi: \Gamma \to W^N$ is a one-to-one mapping. Let μ be a T-invariant measure on $\{0,1\}^N$ such that $\mu(\Gamma) > 0$. Define a measure μ_Γ on Γ by setting

$$\mu_\Gamma(S) = \frac{\mu(S)}{\mu(\Gamma)} \tag{4.23}$$

for any meausrable set $S \subset \Gamma$. Let $\Delta = \{\beta \in \{0,1\}^N;\ \sum_{i=0}^{\infty} \beta(i) = \infty\}$. It is clear that T_Γ and ψ are continuous on $\Gamma \cap \Delta$. Since $\mu(\Gamma \cap \Delta) = \mu(\Gamma)$ for any T-invariant measure μ on $\{0,1\}^N$ (cf. (4.14)), T_Γ and ψ are continuous at almost all points with respect to μ_Γ. Moreover, $\psi(T_\Gamma(\beta)) = T(\psi(\beta))$ for any $\beta \in \Gamma$, where in this equality, T represents the shift on W^N. This fact, combined with the fact that ψ is one-to-one, shows that

$$h_{\mu_\Gamma}(T_\Gamma) = h_{\mu_\Gamma \circ \psi^{-1}}(T).$$

On the other hand, it is known [1] that

$$h_{\mu_\Gamma}(T_\Gamma) = \frac{1}{\mu(\Gamma)} h_\mu(T).$$

Therefore, we have

$$h_{\mu_\Gamma \circ \psi^{-1}}(T) = \frac{1}{\mu(\Gamma)} h_\mu(T). \tag{4.24}$$

LEMMA 4.2. *For $\beta \in \Gamma$, assume that $\overline{\lim}_{n\to\infty} 1/n \sum_{i=0}^{n-1} \beta(i) > 0$. Let $S \in \Xi_\beta$ and $\mu = \mu_\beta^S$. Let $S' = \{\sum_{i=0}^{n-1} \beta(i);\ n \in S\}$. Then, we have $S' \in \Xi_{\psi(\beta)}$ and $\mu_{\psi(\beta)}^{S'} = \mu_\Gamma \circ \psi^{-1}$. Moreover, if β is completely deterministic, then $\psi(\beta)$ is completely deterministic (note that $\mu(\Gamma) > 0$).*

PROOF. Let $f \in C(W^N)$. Define a function g on $\{0,1\}^N$ by

$$g(\gamma) = \begin{cases} f(\psi(\gamma)) & \text{if} \quad \gamma \in \Gamma \\ 0 & \text{else.} \end{cases} \tag{4.25}$$

Then, g is continuous on $\Delta \cup \{0\}$; at almost all points of $\{0,1\}^N$ with respect to μ. For $n \in N$, let $u_n = \sum_{i=0}^{n-1} \beta(i)$. Then,

$$\int (f \circ \psi) d\mu_\Gamma = \frac{1}{\mu(\Gamma)} \int g d\mu$$

$$= \frac{1}{\mu(\Gamma)} \lim_{\substack{n \in S \\ n \to \infty}} \frac{1}{n} \sum_{i=0}^{n-1} g(T^i \beta)$$

(4.26)
$$= \frac{1}{\mu(\Gamma)} \lim_{\substack{n \in S \\ n \to \infty}} \frac{1}{n} \sum_{i=0}^{u_n - 1} f(\psi(T_\Gamma^i \beta))$$

$$= \lim_{\substack{n \in S' \\ n \to \infty}} \frac{1}{n} \sum_{i=0}^{n-1} f(\psi(T_\Gamma^i \beta))$$

$$= \lim_{\substack{n \in S' \\ n \to \infty}} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i \psi(\beta)).$$

Thus, we have $S' \in \Xi_{\psi(\beta)}$ and $\mu_{\psi(\beta)}^{S'} = \mu_\Gamma \circ \psi^{-1}$. Assume that β is completely deterministic. For any $S' \in \Xi_{\psi(\beta)}$, there exists a subset S of the set

$$\left\{ n;\ \sum_{i=0}^{n-1} \beta(i) \in S' \right\}$$

such that $S \in \Xi_\beta$. Since $\{\sum_{i=0}^{n-1} \beta(i);\ n \in S\} \subset S'$, it follows from (4.24) and the fact that $h_{\mu_\beta^S}(T) = 0$ that $h_{\mu_{\psi(\beta)}^{S'}}(T) = 0$. Thus, $\psi(\beta)$ is completely deterministic.

For $\xi = (\xi_0, \xi_1, \cdots, \xi_{q-1}) \in \{0,1\}^q$, denote

(4.27)
$$\chi_\xi(\gamma) = \begin{cases} 1 & \text{if } \gamma(i) = \xi_i \text{ for } i = 0, 1, \cdots, q-1 \\ 0 & \text{else.} \end{cases}$$

LEMMA 4.3. *Let $\alpha \in \mathrm{Nor}_P$. Assume that $\beta \in \{0,1\}^N$ is completely deterministic, and $\underline{\lim}_{n\to\infty} 1/n \sum_{i=0}^{n-1} \beta(i) > 0$. Then, for any infinite subset S' of N, there exists a subset S of S' such that*

(4.28)
$$\lim_{\substack{n \in S \\ n \to \infty}} \frac{1}{n} \sum_{i=0}^{n-1} f(T^{u_i} \alpha) \chi_\xi(T^i \beta) = \int f dP^N \cdot \int \chi_\xi d\mu_\beta^S$$

for any $f \in C(\Sigma^N)$ and $\xi \in D$, where $u_i = \sum_{j=0}^{i-1} \beta(j)$ and

(4.29)
$$D = \bigcup_{q=1}^{\infty} \{(\xi_0, \cdots, \xi_{q-1}) \in \{0,1\}^q : \xi_i = 1 \text{ for some } i = 0, 1, \cdots, q-1\}.$$

PROOF. For $\gamma \in \{0,1\}^N$, define $\hat{\psi}(\gamma) \in W^N$ and $\hat{T}_\Gamma(\gamma) \in \Gamma$ by $\hat{\psi}(\gamma) = \psi(\gamma')$ and $\hat{T}_\Gamma(\gamma) = T_\Gamma(\gamma')$, respectively, where

$$\gamma'(i) = \begin{cases} \gamma(i) & \text{if } i \geqq 1 \\ 1 & \text{if } i = 0. \end{cases}$$

There exists a subset S'' of $\{u_i;\, i \in S'\}$ such that $S'' \in \Xi_{(\alpha,\psi(T^k\beta))}$ for any $k \in N$. Also there exists a subset S of $\{i;\, u_i \in S''\}$ such that $\lim_{n \in S, n \to \infty} u_n/n = b$ exists and $b > 0$. Clearly, $S \subset S'$. Let $\xi = (\xi_0, \cdots, \xi_{q-1}) \in \{0,1\}^q$. First, assume that $\xi_0 = 1$. Then, for any $f \in C(\Sigma^N)$ and $k \in N$, we have

$$\lim_{\substack{n \in S \\ n \to \infty}} \frac{1}{n} \sum_{i=0}^{n-1} f(T^{u_i}\alpha)(\chi_\xi \circ T^k)(T^i\beta)$$

$$= \lim_{\substack{n \in S \\ n \to \infty}} \frac{1}{n} \sum_{i=0}^{n-1} f(T^{u_i}\alpha)\chi_\xi(T^i(T^k\beta))$$

$$= \lim_{\substack{n \in S \\ n \to \infty}} \frac{1}{n} \sum_{i=0}^{u_n} f(T^i\alpha)\chi_\xi(\hat{T}^i_\Gamma(T^k\beta))$$

$$= b \lim_{\substack{n \in S'' \\ n \to \infty}} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i\alpha)\chi_\xi(\hat{T}^i_\Gamma(T^k\beta))$$

$$= b \lim_{\substack{n \in S'' \\ n \to \infty}} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i\alpha)(\chi_\xi \circ \psi^{-1})(T^i\hat{\psi}(T^k\beta))$$

$$= b \int f \cdot (\chi_\xi \circ \psi^{-1})\, d\mu^{S''}_{(\alpha,\psi(T^k\beta))}$$

$$= b \int f \cdot (\chi_\xi \circ \psi^{-1})\, d(P^N \times \mu^{S''}_{\psi(T^k\beta)}) \qquad \text{(from Lemma 3.2 and 4.2)}$$

$$= b \int f dP^N \cdot \int (\chi_\xi \circ \psi^{-1})\, d\mu^{S''}_{\psi(T^k\beta)}$$

$$= b \int f dP^N \cdot \int \chi_\xi\, d(\mu^S_{T^k\beta})_\Gamma \qquad \text{(from Lemma 4.2)}$$

$$= \int f dP^N \cdot \int \chi_\xi\, d\mu^S_{T^k\beta}$$

$$= \int f dP^N \cdot \int \chi_\xi\, d\mu^S_\beta.$$

Next, let $\xi_0 = \cdots = \xi_{s-1} = 0$ and $\xi_s = 1$ for some $1 \leqq s \leqq q-1$. Then, we have

$$\chi_\xi = \chi_{(\xi_s,\dots,\xi_{q-1})} \circ T^s - \sum_{k=0}^{s-1} (1, \underbrace{0, \cdots, 0,}_{k} \xi, \cdots, \xi_{q-1}) \circ T^{s-k-1}$$

Thus, (4.28) follows from the first case.

PROOF OF (4.9). Let us assume (4.10) and Condition 4. It is sufficient to prove that $\mathrm{Nor}_P \circ \tau \supset \mathrm{Nor}_P$, since Lemma 4.1. Let

$$\{d_0 < d_1 < \cdots\} = \{\tau(i);\, i \in N\}^c. \tag{4.30}$$

Let $X = (X_0, X_1, \cdots)$ be an independent sequence of Σ-valued random variables such that the distribution of X_i on Σ is P for each $i \in N$. Let $\alpha \in \mathrm{Nor}_P$ be arbitrary. Let

$$Y(i) = \begin{cases} \alpha(j) & \text{if } i = \tau(j) \\ X_j & \text{if } i = d_j \end{cases} \qquad (i \in N). \tag{4.31}$$

Then, clearly $Y \circ \tau = \alpha$. Let F be a countable base of $C(\Sigma^N)$ such that $F \subset \bigcup_{q=0}^{\infty} C(\Sigma^q)$. Let $f \in C(\Sigma^q)$. For $\xi = (\xi_0, \cdots, \xi_{q-1}) \in \{0,1\}^q$, define $f_\xi \in C(\Sigma^N)$ by

$$f_\xi(\gamma) = \int f d(\lambda^{\xi_0}_{\gamma(b_0)} \times \cdots \times \lambda^{\xi_{q-1}}_{\gamma(b_{q-1})}) \qquad \left(b_i = \sum_{j=0}^{i-1} \xi_j,\, i = 0, 1, \cdots, q-1\right), \tag{4.32}$$

where $\lambda^0_a = P$ and $\lambda^1_a = \delta_a$ for any $a \in \Sigma$. Then, it is clear that $\int f_\xi \, dP^N = \int f dP^N$ for any $\xi \in \{0,1\}^q$. It is also clear that

$$E\{f(T^i Y)\} = \sum_{\xi \in [0,1]^q} f_\xi(T^{u_i}\alpha)\chi_\xi(T^i\theta_\tau), \tag{4.33}$$

where $u_i = \sum_{j=0}^{i-1} \theta_\tau(j)$. Note that if $\xi = (0, 0, \cdots, 0)$, then f_ξ is a constant function equal to $\int f dP^N$. Therefore, it follows from Lemma 4.3 that for any infinite subset S' of N, there exists a subset S of S' such that

$$\lim_{\substack{n \in S \\ n \to \infty}} \frac{1}{n} \sum_{i=0}^{n-1} E\{f(T^i Y)\} = \sum_{\xi \in [0,1]^q} \int f dP^N \cdot \int \chi_\xi d\mu^S_{\theta_\tau} = \int f dP^N.$$

This fact implies that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} E\{f(T^i Y)\} = \int f dP^N. \tag{4.34}$$

Note that the strong law of large number holds for the sequence of random variables $(f(T^i Y);\, i = 0, 1, \cdots)$. Therefore, (4.34) implies that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i Y) = \int f dP^N$$

holds with probability 1. Since F is a countable set, we can take $\gamma \in \Sigma^N$ such that $\gamma \circ \tau = \alpha$ and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i \gamma) = \int f dP^N$$

for any $f \in F$. Since F is a base of $C(\Sigma^N)$, it holds that $\gamma \in \mathrm{Nor}_P$. Thus, $\alpha \in \mathrm{Nor}_P \circ \tau$ which completes the proof of (4.9).

In conclusion, we have proved the following result.

THEOREM 4. *Under the hypothesis* $\overline{\lim}_{n\to\infty} \tau(n)/n < \infty$, *Conditions* 1, 2, 3 *and* 4 *are equivalent to each other.*

REMARK. It can be proved that if $\alpha \in \Sigma^N$ is a stochastic sequence such that the endomorphism T on (Σ^N, μ_α) has a completely positive entropy, or equivalently, the natural extension of it is a Kolmogorov automorphism, then α is a τ-collective for any τ such that $\overline{\lim}_{n\to\infty} \tau(n)/n < \infty$ and θ_τ is completely deterministic.

5. Completely deterministic 0-1 sequences

The notion of completely deterministic sequence is obviously extended to a more general case if the base space $\{0,1\}$ is replaced by a compact metric space Σ That is, $\alpha \in \Sigma^N$ is said to be *completely deterministic* if $h_\mu(T) = 0$ for any $\mu \in \{\mu_\alpha^S; S \in \Xi_\alpha\}$.

EXAMPLE 1. The following types of sequences are known to be completely deterministic:

1. Toeplitz type sequences [5]
2. generalized Morse sequences [8]
3. sequences associated with substitutions, [4] or [7]
4. sequences generated by finite automata in the sense of [6].

For α and β belonging to $\{0,1\}^N$, define $\alpha \leftarrow \beta$ and $\alpha * \beta$ belonging to $\{0,1\}^N$ by

$$(\alpha \leftarrow \beta)(i) = \begin{cases} 0 & \text{if } \alpha(i) = 0 \\ \beta\left(\sum_{j=0}^{i-1} \alpha(j)\right) & \text{if } \alpha(i) = 1, \text{ and} \end{cases}$$

$$(\alpha * \beta)(i) = \begin{cases} \alpha(i) & \text{if } \beta(i) = 1 \\ 0 & \text{if } \beta(i) = 0. \end{cases}$$

Also, for $\alpha \in \{0,1\}^N$ and $\beta \in \{0,1\}^N$ such that $\Sigma_{i=0}^{\infty} \beta(i) = \infty$, define $\alpha | \beta \in \{0,1\}^N$ by

$$(\alpha | \beta)(i) = \alpha\left(\min\left\{j; \sum_{k=0}^{j} \beta(k) = i + 1\right\}\right).$$

Let α and β belong to $\{0,1\}^N$. Let τ and κ be selection functions. Then, it is clear that

$$\theta_\tau \leftarrow \theta_\kappa = \theta_{\tau \circ \kappa}, \text{ and} \tag{5.1}$$
$$\alpha \,|\, \theta_\tau = \alpha \circ \tau.$$

LEMMA 5.1. *If α and β are completely deterministic, then $\alpha * \beta$ is completely deterministic.*

PROOF. Let ψ be a mapping $\{0,1\}^N \times \{0,1\}^N \to \{0,1\}^N$ such that $\psi(\gamma,\delta) = \gamma * \delta$. Then, it is clear that ψ is a continuous mapping such that $\psi \circ T = T \circ \psi$. Therefore, for any $S \in \Xi_{\alpha*\beta}$, it holds that

$$h_{\mu^S_{\alpha*\beta}}(T) = h_{\mu^S_{(\alpha,\beta)}\circ\psi^{-1}}(T) \leqq h_{\mu^S_{(\alpha,\beta)}}(T) \leqq h_{\mu^S_\alpha}(T) + h_{\mu^S_\beta}(T),$$

from which Lemma 5.1 follows.

LEMMA 5.2. *Assume that $\sum_{i=0}^{\infty}\beta(i) = \infty$. Then,*

$$\beta \leftarrow (\alpha \,|\, \beta) = \alpha * \beta.$$

PROOF. Clear.

Let us introduce the following condition about $\gamma \in \{0,1\}^N$;

$$\varliminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \gamma(i) > 0. \tag{5.2}$$

The following theorem follows from Theorem 4.

THEOREM 5. *Let τ and κ be selection functions satisfying* (4.10). *Assume that θ_τ is completely deterministic. Then, $\theta_{\tau\circ\kappa}$ is completely deterministic if and only if θ_κ is completely deterministic.*

COROLLARY. *Let α and β be 0–1sequences satisfying* (5.2). *Assume that α is completely deterministic. Then, $\alpha \leftarrow \beta$ is completely deterministic if and only if β is completely deterministic.*

The following theorem follows from Theorem 5, Lemma 5.1 and 5.2.

THEOREM 6. *Let α be a 0–1sequence and τ be a selection function such that each of θ_τ and $\alpha \circ \tau$ satisfies* (5.2). *Assume that α and θ_τ are completely deterministic. Then, $\alpha \circ \tau$ is completely deterministic.*

COROLLARY.*Let α and β be 0–1 sequences such that each of β and $\alpha \,|\, \beta$ satisfies* (5.2). *Assume that α and β are completely deterministic. Then, $\alpha|\beta$ is completely deterministic.*

EXAMPLE 2. Let b and c be real numbers such that $b \geqq 1$ and $c \geqq 0$. Let $\tau(i) = [bi + c]$. Then, θ_τ is completely deterministic [14]. Therefore, from Theorem 5, θ_τ is completely deterministic if

$$\tau(i) = [b_k[\cdots[b_2[b_1 i + c_1] + c_2]\cdots] + c_k],$$

where $b_1, \cdots, b_k$ are real numbers $\geqq 1$ and $c_1, \cdots, c_k$ are real numbers $\geqq 0$.

REMARK. One can prove the "if" part of Theorem 5 direetly without using Theorem 4. The author was informed of the idea of the direct proof by Professor Benjamin Weiss in a letter.

ACKNOWLEDGEMENTS

The author wishes to thank Professor Takesi Watanabe for frequent discussions which were helpful for producing the results written here and for useful advice and suggestions. The proof of Lemma 4.1 was shortened by virtue of his suggestion. The author also wishes to thank Professor Nobuyuki Ikeda and Professor Haruo Totoki for useful discussions on the subject.

REFERENCES

1. L. M. Abramov, *The entropy of the derived automorphism*, Dokl. Akad. Nauk. SSSR **128** (1959), 647–650.

2. P. Billingsley, *Ergodic Theory and Information*, John Wiley and Sons, New York, 1965.

3. H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory 1-1 (1967), 1–49.

4. W. H. Gottschalk, *Substitution minimal sets*, Trans. Amer. Math. Soc. **109** (1963), 467–491.

5. K. Jacobs and M. Keane, 0–1-*Sequences of Toeplitz type*, *Z.* Wahrscheinlichkeitstheorie Verw. Gebiete **13** (1969), 123–131.

6. Teturo Kamae, *Spectrum of a substitution minimal set*, J. Math. Soc. Japan **22** (1970), 567–578.

7. Teturo Kamae, *A topological invariant of substitution minimal sets*, J. Math. Soc. Japan **24** (1972). 285–306.

8. M. Keane, *Generalized Morse sequences*, Z. Wahrscheinlichkeitstheorie Verw. Gebiete **10** (1968), 335–353.

9. J. C. Oxtoby, *Ergodic sets*, Bull. Amer. Math. Soc. **58** (1961), 116–136.

10. J. C. Oxtoby, *On two theorems of Parthasarathy and Kakutani concerning the shift transformation*, Ergodic Theory, Fred B. Wright (ed.), Academic Press, New York, 1962.

11. K. R. Parthasarathy, *Probability Measures on Metric Spaces*, Academic Press, New York, 1967.

12. V. A. Rokhlin, *New progress in the theory of transformations with invariant measure*, Russian Math. Surveys **15** (1960), 1–22.

13. H. Totoki, *Introduction to Ergodic Theory*, Kyoritsu Shuppan, Tokyo, 1971 (in Japanese).

14. B. Weiss, *Normal sequences as collectives*, Proc. Symp. on Topological Dynamics and Ergodic Theory, Univ. of Kentucky, 1971.

DEPARTMENT OF MATHEMATICS
OSAKA CITY UNIVERSITY
SUGIMOTO-CHO, OSAKA, JAPAN